

POROSITY OF THE BRANCH SET OF DISCRETE OPEN MAPPINGS WITH CONTROLLED LINEAR DILATATION

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Dedicated to Juha Heinonen with admiration and appreciation

ABSTRACT. Assume that X and Y are locally compact and locally doubling metric spaces, which are also generalized n -manifolds, that X is locally linearly locally n -connected, and that Y has bounded turning. Let $f: X \rightarrow Y$ be a continuous, discrete and open mapping. Let \mathcal{B}_f be the branch set of f , i.e. the set consisting of points in X at which f fails to be a local homeomorphism.

In this paper, addressing Heinonen's ICM 02 talk, we study the geometry of the branch set \mathcal{B}_f of a quasiregular mapping between metric n -manifolds. In particular, we show that $\mathcal{B}_f \cap \{x \in X : H_f(x) < \infty\}$ is countably porous, as is its image $f(\mathcal{B}_f \cap \{x \in X : H_f(x) < \infty\})$. As a corollary, $\mathcal{B}_f \cap \{x \in X : H_f(x) < \infty\}$ and its image are null sets with respect to any locally doubling measures on X and Y , respectively. Moreover, if either $H_f(x) \leq H$ or $H_f^*(x) \leq H^*$ for all $x \in X$, then both \mathcal{B}_f and $f(\mathcal{B}_f)$ are countably δ -porous, quantitatively, with a computable porosity constant.

When further metric and analytic assumptions are placed on X , Y , and f , our theorems generalize the well-known Bonk–Heinonen theorem and Sarvas' theorem to a large class of metric spaces. Moreover, our results are optimal in terms of the underlying geometric structures. As a direct application, we obtain the important Väisälä's inequality in greatest generality. Applying our main results to special cases, we solve an open problem of Heinonen–Rickman, and an open question of Heinonen–Semmes.

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1. INTRODUCTION

A continuous mapping $f: X \rightarrow Y$ between topological spaces is said to be a *branched covering* if f is *discrete* and *open*, i.e. f is an open mapping and for each $y \in Y$ the preimage $f^{-1}(y)$ is a discrete subset of X . The *branch set* \mathcal{B}_f of f is the closed set of points in X where f does not define a local homeomorphism. In the case that X and Y are *generalized n -manifolds*, \mathcal{B}_f can be interpreted alternatively as the set of points at which the *local index* $i(x, f) = 1$.

By a result of due to Chernavskiĭ and Väisälä [44], *the branch set \mathcal{B}_f of a branched cover $f: X \rightarrow Y$ between n -manifolds has topological dimension at most $n - 2$* . In dimension $n \geq 5$, there are branched coverings between n -manifolds with branch set of topological dimension $n - 4$. It is not known, however, whether the topological dimension of the branch set of a branched cover between two 3-manifolds is 1.

It should be noticed that even for a branched covering $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, both $\dim_{\mathcal{H}}(\mathcal{B}_f)$ and $\dim_{\mathcal{H}} f(\mathcal{B}_f)$ can be equal to n . Thus towards Hausdorff dimensional estimates of the branch set \mathcal{B}_f , further analytic assumptions have to be imposed on the branched coverings $f: X \rightarrow Y$. The common classes of mappings that arose great interests in the past two decades are the so-called *quasiregular mappings*, or *mappings of bounded distortion*; see [2, 28, 29, 30, 36] for the general theory of quasiregular mappings.

For a branched covering $f: X \rightarrow Y$ between two metric spaces, $x \in X$ and $r > 0$, set

$$H_f(x, r) = \frac{L_f(x, r)}{l_f(x, r)},$$

where

$$L_f(x, r) := \sup \{d(f(x), f(y)) : d(x, y) = r\},$$

and

$$l_f(x, r) := \inf \{d(f(x), f(y)) : d(x, y) = r\}.$$

Then the *linear dilatation function* of f at x is defined pointwise by

$$H_f(x) = \limsup_{r \rightarrow 0} H_f(x, r).$$

A mapping $f: X \rightarrow Y$ between two metric measure spaces is termed (*metrically*) H -*quasiregular* if the linear dilatation function H_f is finite everywhere and essentially bounded from above by H . We call f a quasiregular mapping if it is H -quasiregular for some $H \in [1, \infty)$.

The branch set of a quasiregular mapping can be very wild, for instance, it might contain many wild Cantor sets, such as the Antoine's necklace [20], of classical geometric topology. In his 2002 ICM address [16, Section 3], Heinonen asked the following question: *Can we describe the geometry and topology of allowable branch sets of quasiregular mappings between metric n -manifolds?*

Let us point out that the study of the geometry and topology of the branch set of a quasiregular mapping will lead to numerous important consequences. For instance, a deeper understanding of the geometry of branch set of a quasiregular mapping

- helps in establishing the general theory of quasiregular mappings in non-smooth metric spaces. The principle is the following: it is usually much easier to establish the theory of *quasiconformal mappings*, i.e. injective quasiregular mappings, in general metric setting; the difference between quasiconformal mappings and quasiregular mappings lies in the branch set; negligible branch points and their image do not affect most of the local properties.
- helps to establish quantitative Hausdorff dimensional estimates for the branch set \mathcal{B}_f and its image $f(\mathcal{B}_f)$ of a quasiregular mapping; see for instance [41, 3, 33] and Corollary 1.3 below;
- helps to establish the important Väisälä's inequality in general metric spaces, which is crucial for generalizing the value distributional type results (such as Picard type theorems and defect relation) beyond Euclidean spaces see for instance [40, 33, 47] and Sections 7 below;

In this paper, we explore the (geometric) porosity of $\mathcal{B}_f \cap A$ and $f(\mathcal{B}_f \cap A)$ when the linear dilatation of f is finite on A . Our main result states that if X satisfies a quantitative local connectivity assumption, the aforementioned sets are quantitatively porous. As mentioned earlier, this leads to quantitative Hausdorff dimensional estimates of these sets.

Regarding the Hausdorff dimension of \mathcal{B}_f and its image $f(\mathcal{B}_f)$ in the Euclidean setting, a well-known result of Gehring and Väisälä [7] says that for each $n \geq 3$ and each pair of numbers $\alpha, \beta \in [n-2, n)$, there exists a quasiregular mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\dim_{\mathcal{H}} \mathcal{B}_f = \alpha \quad \text{and} \quad \dim_{\mathcal{H}} f(\mathcal{B}_f) = \beta.$$

On the other hand, by the result of Sarvas [41], for a non-constant H -quasiregular mapping $f: \Omega \rightarrow \mathbb{R}^n$, $n \geq 2$, between Euclidean domains,

$$(1.1) \quad \dim_{\mathcal{H}} f(\mathcal{B}_f) \leq n - \eta$$

for some constant $\eta = \eta(n, H) > 0$.

It has been an open problem for a long time whether the analogous dimensional estimate holds also for the branch set \mathcal{B}_f . The answer turns out to be yes, as a well-known

result of Bonk and Heinonen [3] says that for a non-constant H -quasiregular mapping $f: \Omega \rightarrow \mathbb{R}^n$, $n \geq 2$, between Euclidean domains,

$$(1.2) \quad \dim_{\mathcal{H}} \mathcal{B}_f \leq n - \eta$$

for some constant $\eta = \eta(n, H) > 0$. Let us point out that the result of Bonk and Heinonen [3] relies on an earlier theorem of Sarvas [41], which implies the existence of a quantitative upper bound on $\dim_{\mathcal{H}}(\{x \in \mathbb{R}^n : 2 \leq i(x, f) \leq m\})$ below n , depending only on K , n , and m . The Bonk–Heinonen theorem then follows upon proving that $\{x \in \mathbb{R}^n : i(x, f) > m\}$ is porous, for some m depending only on n and K .

Since the Sarvas theorem used a normal family argument, the dimension bound obtained in (1.2) was not directly computable; Onninen and Rajala [33] later proved that the sets $\{x \in \mathbb{R}^n : i(x, f) \leq m\}$ are δ_m -porous, as are their images under f , with a directly computable porosity constant δ_m , which combined with the Bonk–Heinonen porosity result for points of large index, gives a computable dimension bound on $\dim_{\mathcal{H}} \mathcal{B}_f$.

On the other hand, very little is known about the Hausdorff dimension of \mathcal{B}_f and its image $f(\mathcal{B}_f)$ for quasiregular mapping $f: X \rightarrow Y$ beyond the Euclidean spaces. Indeed, in the non-smooth setting, all the known proofs of the fact that both the branch set and its image are null sets with respect to an Ahlfors regular measure relies on certain (Lip-schitz) differentiable structure akin to the Euclidean spaces. For instance, Heinonen and Rickman [21] have established a general theory of *mappings of bounded length distortion* (BLD mappings for short), which form a proper subclass of quasiregular mappings, between the so-called *generalized manifolds of type A*, on which both the branch set and its image are null sets with respect to the Ahlfors regular measures, and put it as an open problem [21, Remark 6.7 (b)] whether it is possible to obtain a quantitative estimate as (1.2). Note that a quantitative estimate as (1.2) can be applied to improve on the main result of [23], regarding the size of the exceptional set of the bi-Lipschitz parametrization. For quasiregular mappings from the Euclidean domain to generalized manifolds of type A, Onninen and Rajala [33] were able to obtain a slightly weaker estimate of the form (1.1), while an estimate of the form (1.2) seems to be un-reachable.

In the remainder of this introduction, we take as standing assumptions that X and Y are locally compact and locally doubling metric spaces, which are also generalized n -manifolds, that X is locally linearly locally n -connected, and that Y has bounded turning (precise definitions are given in Sections 2 and 4). We also assume throughout this paper that $f: X \rightarrow Y$ is continuous, discrete and open.

For each $R > 0$ and $H \geq 1$, set

$$S_{H,R} = \{x \in X : H_f(x, r) \leq H \text{ for all } r < R\}$$

and

$$S_{H,R}^* = \{x \in X : H_f^*(x, r) \leq H \text{ for all } r < R\},$$

where H_f^* is the inverse dilatation function as defined in Section 2.2. Then we denote

$$S_H = \cup_{R>0} S_{H,R}, \quad S_H^* = \cup_{R>0} S_{H,R}^*,$$

and

$$S_f = \cup_{H < \infty} S_H, \quad S_f^* = \cup_{H < \infty} S_H^*.$$

Our main result says that under these assumptions, most points where the dilatation or inverse dilatation is finite are not branch points.

Theorem 1.1. For each $R > 0$ and $H \geq 1$, the set $S_{H,R} \cap \mathcal{B}_f$, $f(S_{H,R} \cap \mathcal{B}_f)$, $S_{H,R}^* \cap \mathcal{B}_f$ and $f(S_{H,R}^* \cap \mathcal{B}_f)$ are δ -porous, where δ depends only on H and the data of X and Y . In particular, if $H_f(x) \leq H$ or $H_f^*(x) \leq H$ for every $x \in X$, then \mathcal{B}_f and $f(\mathcal{B}_f)$ are countably δ -porous, quantitatively. Moreover, the porosity constant can be explicitly calculated.

In the special case that X and Y are Euclidean spaces, Theorem 1.1 gives a nice decomposition of the the branch set of a branched covering into countable union of sets restricted to which the branch set is quantitatively porous. Thus, it can be regarded as a strengthened version of the earlier quantitative porosity results of Bonk–Heinonen [3] and Onninen–Rajala [33] for the branch set of a quasiregular mapping. Moreover, the quantitative porosity bounds on $f(\mathcal{B}_f)$ seems to be new even for a quasiregular mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and it can be regarded as a strengthened version of the dimensional estimate of Sarvas [41].

Corollary 1.2. For all locally doubling measures μ on X and ν on Y ,

$$\mu(S_f \cap \mathcal{B}_f) = \nu(f(S_f \cap \mathcal{B}_f)) = \mu(S_f^* \cap \mathcal{B}_f) = \nu(f(S_f^* \cap \mathcal{B}_f)) = 0.$$

In particular, if either $H_f(x) < \infty$ or $H_f^*(x) < \infty$ for all $x \in X$, then

$$\mu(\mathcal{B}_f) = \nu(f(\mathcal{B}_f)) = 0.$$

If $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a *mapping of finite linear dilatation* (i.e. f is a branched covering and satisfies $H_f(x) < \infty$ for almost everywhere $x \in \mathbb{R}^n$) with locally exponentially integrable linear dilatation (i.e. $\exp(\lambda H_f) \in L_{\text{loc}}^1(\mathbb{R}^n)$ for some positive constant λ), then it follows from the earlier works of Kallunki [26, Theorem 4.5] and Koskela–Malý [27, Theorem 1.1] that \mathcal{B}_f is a null set with respect to the n -dimensional Lebesgue measure. Somewhat surprisingly, Corollary 1.2 implies that the assumption that f has a locally exponentially integrable linear dilatation is superfluous.

Particularly important to the general theory of quasiconformal and quasisymmetric mappings are Ahlfors Q -regular spaces. It is well know that porous subsets of such spaces have Hausdorff dimension strictly smaller than Q , quantitatively; see e.g. [4, Lemma 3.12] or [33, Lemma 9.2]. Thus we have the following consequence.

Corollary 1.3. If X and Y are Ahlfors Q -regular, and $H_f(x) < \infty$ or $H_f^*(x) < \infty$ for all $x \in X$, then $\mathcal{H}^Q(\mathcal{B}_f) = \mathcal{H}^Q(f(\mathcal{B}_f)) = 0$. Moreover, if either $H_f(x) \leq H$ or $H_f^*(x) \leq H$ for all $x \in X$, then

$$\max \{ \dim_{\mathcal{H}}(\mathcal{B}_f), \dim_{\mathcal{H}}(f(\mathcal{B}_f)) \} \leq Q - \eta < Q,$$

where η depends only on H and the data of X and Y . Moreover, η can be explicitly calculated.

Our methods are closest in spirit to those of Onninen and Rajala [33]. The principle differences are three-fold: firstly, we circumvent the need for analytic arguments based on modulus inequalities, instead arguing directly from the infinitesimal metric definition. Thus we avoid the need for analytic assumptions on the domain, e.g., the Poincaré inequality. This is not completely surprising -in the special setting that the Hausdorff and topological dimensions of X coincide, i.e. $Q = n$, a deep result of Semmes [42, Theorem B.10] states that *linear local contractibility* implies an abstract Poincaré inequality in the sense of Heinonen–Koskela [18]. On the other hand, this clearly fails for $Q > n$ since we may snowflake the space so that there are no rectifiable curves. As far as we know, this is the first case where the estimates on the branch set and its image are obtained when $Q \neq n$. Moreover, it is quite surprising that properties other than differentiability of quasiregular mappings can be deduced directly from the metric definition, which is often difficult to use because of the infinitesimal feature.

The second substantial difference from [33] is that their methods depend on a theorem of McAuley–Robinson [32] giving a lower bound on the diameter of certain point inverses for nonhomeomorphic discrete open mappings with Euclidean domains. The argument in [32] depends crucially on the affine structure of the Euclidean spaces; to generalize it to our setting, we require methods from quantitative topology as developed by Grove, Petersen, Wu and Semmes [12, 13, 34, 42].

The third major difference from [33] is that, instead of splitting the branch points into two parts -one with large local index and the other with bounded local index- we argue directly on the branch set $\mathcal{B}_f \cap S_{H,R}$ and $f(\mathcal{B}_f \cap S_{H,R})$, and so our estimates on these sets are automatically index-free. In particular, when the underlying metric spaces are Loewner, H_f^* will be quantitatively bounded and thus the branch set of a quasiregular mapping can be decomposed into a countable union of porous sets with quantitative porosity constant. It is worth pointing out that our method allows us to obtain quantitative countable porosity bounds for both the branch set \mathcal{B}_f and its image $f(\mathcal{B}_f)$, simultaneously.

1.1. Analytic consequences. When X and Y are Ahlfors Q -regular, finiteness and essential boundedness of either $H_f(x)$ or $H_f^*(x)$ (or, for that matter, of even one of the “lim inf”- dilatations $h_f(x)$ or $h_f^*(x)$ in the spirit of [17, 1]) implies that on each open set $U \subset X$, the K_O - and K_I -inequalities

$$\frac{1}{K_O N_f(U)} \text{Mod}_Q(\Gamma) \leq \text{Mod}_Q(f(\Gamma)) \leq K_I \text{Mod}_Q(\Gamma)$$

hold for every family Γ of curves in X , where $N_f(U) = \sup_{y \in Y} \text{card}(f^{-1}(y) \cap U)$. This was proved in the homeomorphic case in [46, Theorem 1.6] and later extended to the branched setting in [47] (see [14]). Neither of these inequalities require any assumptions on local homology or contractibility for X and Y . It is also shown in [47] (see [14]) that whenever $\mathcal{H}^Q(f(\mathcal{B}_f)) = 0$, the K_I -inequality is equivalent to the typically stronger Väisälä’s inequality, given precisely in Theorem 7.3. Thus the first part of Corollary 1.3 gives Väisälä’s inequality in our setting, provided H_f or H_f^* is finite and essentially bounded

(see Theorem 7.3 below). At this point, we have still not imposed any Poincaré inequality on X or Y .

1.2. Loewner spaces. There is a subtlety to the observation that Corollary 1.3 generalizes the Bonk–Heinonen theorem, which gave an index-free upper bound on $\dim_{\mathcal{H}} \mathcal{B}_f$. In general, the linear dilatation $H_f(x)$ of a quasiregular map in \mathbb{R}^n need not be globally bounded - it is instead finite and essentially bounded, and at any point $x \in \mathbb{R}^n$, the dilatation depends quantitatively on not merely the essential supremum of H_f , but also on the index $i(x, f)$. That Corollary 1.3 is an actual generalization requires the fact that $H_f^*(x)$ is bounded everywhere by a constant H^* independent of $i(x, f)$. This latter fact was proved in the Euclidean case in [30], using the K_O - and Väisälä's inequalities, as well as the Loewner property of \mathbb{R}^n . Thus we do not know, in the Q -regular case, whether finiteness and essential boundedness of H_f is sufficient to obtain an upper bound for $\dim_{\mathcal{H}} \mathcal{B}_f$ (nor, for that matter, for $\dim_{\mathcal{H}} f(\mathcal{B}_f)$).

In the case that X and Y are Loewner, however, Väisälä's inequality allows us to generalize the corresponding result of [30], giving an index free upper bound on H_f^* .

Theorem 1.4. Suppose (under the standing assumptions) that X and Y are locally Ahlfors Q -regular and Q -Loewner, $H_f(x) < \infty$ for all $x \in X$, and $H_f(x) \leq H$ for \mathcal{H}^Q -almost every $x \in X$. Then $H_f^*(x) \leq H^*$ for every $x \in X$, where H^* depends only on H and the data of X and Y , and the sets \mathcal{B}_f and $f(\mathcal{B}_f)$ are countably δ -porous, for some δ depending only on H and the data.

Combining Theorem 1.4 with Corollary 1.3, we obtain the following result, the first half of which is a true generalization of the Bonk-Heinonen theorem.

Corollary 1.5. Under the assumptions of Theorem 1.4, we have

$$\max \{ \dim_{\mathcal{H}}(\mathcal{B}_f), \dim_{\mathcal{H}}(f(\mathcal{B}_f)) \} \leq Q - \eta < Q,$$

for some constant η depending only on H and the data of X and Y .

Corollary 1.5 answers affirmatively the open problem of Heinonen and Rickman [21, Remark 6.7 (b)] in a stronger form, namely, we obtain dimensional estimates for the class of quasiregular mappings, which is strictly large than the class of BLD mappings¹. Notice also that we have obtained the dimensional estimates for both the branch set and its image.

In [22, Question 27], Heinonen and Semmes asked *if for a given branched covering $f : S^n \rightarrow S^n$, $n \geq 3$, there is a metric d on S^n so that (S^n, d) is an Ahlfors n -regular and locally linearly contractible metric space, and $f : (S^n, d) \rightarrow S^n$ is a BLD mapping*. By Corollary 1.5, the existence of such a metric d necessarily implies that $f(\mathcal{B}_f)$ must be null with respect to the n -dimensional Hausdorff measure \mathcal{H}^n . On the other hand, there are

¹In [21], the problem was asked for mappings between generalized n -manifolds of type A , which do not necessarily have quantitative data as in the setting of the above corollary. However, it is very evident that one needs to impose quantitative data in order to obtain quantitative dimensional estimates on the branch set of quasiregular mappings

plenty of branched coverings $f: S^n \rightarrow S^n$ such that $\mathcal{H}^n(f(\mathcal{B}_f)) > 0$ and so we have the following negative answer to this question.

Corollary 1.6. Not every branched covering $f: S^n \rightarrow S^n$, $n \geq 3$, can be made BLD by changing the metric in the domain but keeping the space Ahlfors n -regular and linearly locally contractible.

1.3. Sharpness of the results. Our standing assumptions for the underlying spaces X and Y , except the local linear n -connectivity on X , are quite mild. On the other hand, the local linear n -connectivity is necessary for the validity of all the previous results, as the following example from [14] indicates.

Theorem 1.7 (Corollary 8.7, [14]). For each $n \geq 3$, there exist an Ahlfors n -regular metric space X that is homeomorphic to \mathbb{R}^n and supports a $(1, 1)$ -Poincaré inequality, and a 1-quasiregular mapping $f: X \rightarrow \mathbb{R}^n$, such that $\min \{ \mathcal{H}^n(\mathcal{B}_f), \mathcal{H}^n(f(\mathcal{B}_f)) \} > 0$.

The construction of such an example, as in Theorem 1.7, is demonstrated in [14, Corollary 8.7], whereas the mapping $f: X \rightarrow \mathbb{R}^n$ is shown to be even 1-BLD.

1.4. Removing the topological assumptions. The assumption that X and Y are generalized manifolds is used in Theorem 1.1 only once, in order to apply our generalization of the McAuley–Robinson theorem, Corollary 4.8. If we remove the local homology assumption, we may still apply Theorem 4.7, to obtain local (left) homotopy inverses away from a porous set. In particular, Theorem 1.1 and Corollaries 1.2 and 1.3 all remain valid if \mathcal{B}_f is replaced with the left homotopy branch set $\mathcal{B}_f^{*,l}$ (see Section 2.5 for the definition), consisting of all the points at which f fails to have a local left homotopy inverse g as given in the conclusion of Theorem 4.7. Thus, when $x \notin \mathcal{B}_f^{*,l}$, the homomorphisms f_* and f^* on local homology and cohomology have left (resp. right) inverses.

It is also not too hard to show under the assumptions of the theorem that the sets U_α constructed in the proof of Theorem 1.1 also satisfy $\text{diam } f(U_\alpha) \ll d(y_0, f(U_\alpha))$, provided δ is sufficiently small.

Thus if Y , as well as X , is assumed to be LLC^n , then Proposition 4.6 may be applied to $f \circ g$ to obtain a homotopy equivalence $f \circ g \simeq I_{B(y_0, r)}$ through which the boundary $\partial B(y_0, r)$ remains far away from y_0 .

Thus we could replace \mathcal{B}_f in Theorem 1.1 and Corollaries 1.2 and 1.3 with a generalized homotopy branch set \mathcal{B}_f^* . We would in particular have that at each $x \notin \mathcal{B}_f^*$, the induced maps f_* and f^* on local homology and cohomology are isomorphisms.

We summarize the above observations as a separate theorem.

Theorem 1.8. Removing the assumption that X and Y are generalized n -manifolds from the standing assumptions, Theorem 1.1 and Corollaries 1.2 and 1.3 remain valid if we replace the branch set \mathcal{B}_f with the generalized left homotopy branch set $\mathcal{B}_f^{*,l}$. Moreover, if Y is additionally assumed to be LLC^n , then all the conclusions hold if we replace the branch set \mathcal{B}_f with the generalized homotopy branch set \mathcal{B}_f^* .

It should be noticed that we require the -LLC^n condition on X to construct local left homotopy inverse and with the additional assumption Y being -LLC^n we may construct

a local (two-sided) homotopy inverse. It is natural to ask what happens if only Y , but not X , is assumed to be linearly locally n -connected.

On the other hand, the example from Theorem 1.7 (see [14, Section 8.3] for the construction) implies that the answer to this question is no. For the branched covering $f: X \rightarrow \mathbb{R}^n$ as in Theorem 1.7, since X and \mathbb{R}^n are topological n -manifolds, $\mathcal{B}_f = \mathcal{B}_f^{*,r}$, whence $\mathcal{B}_f^{*,r}$ and its image are not even null sets with respect to \mathcal{H}^n , let alone being porous, so the analogues to Theorem 1.1 and Corollaries 1.2 and 1.3 all fail.

1.5. Outline of the paper. This paper is organized as follows. Section 1 contains the introduction and Section 2 some preliminaries. In Section 3, we show that BLD mappings are quantitatively quasiregular in a large class of metric spaces. In Section 4, we develop a quantitative ENR theory for linearly locally n -connected spaces. In particular, we obtain a generalized version of the McAuley–Robinson theorem. In Section 5, we obtain quantitative control of the distortion of annuli at points with finite dilatation, away from a porous set. The proofs of our main results are given in Section 6. We establish the important Väisälä’s inequality in Section 7.

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2. PRELIMINARIES

2.1. Generalized manifolds and topological degree. Let $H_c^*(X)$ denote the Alexander–Spanier cohomology groups of a space X with compact supports and coefficients in \mathbb{Z} .

Definition 2.1. A space X is called an n -dimensional, $n \geq 2$, *cohomology manifold* (over \mathbb{Z}), or a *cohomology n -manifold* if

- (a): the cohomological dimension $\dim_{\mathbb{Z}} X$ is at most n , and
- (b): the local cohomology groups of X are equivalent to \mathbb{Z} in degree n and to zero in degree $n - 1$.

Condition (a) means that $H_c^p(U) = 0$ for all open $U \subset X$ and $p \geq n + 1$. Condition (b) means that for each point $x \in X$, and for each open neighborhood U of x , there is another open neighborhood V of x contained in U such that

$$H_c^p(V) = \begin{cases} \mathbb{Z} & \text{if } p = n \\ 0 & \text{if } p = n - 1, \end{cases}$$

and the standard homomorphism

$$(2.1) \quad H_c^n(W) \rightarrow H_c^n(V)$$

is a surjection whenever W is an open neighborhood of x contained in V . As for examples of cohomology n -manifolds, we point out all topological n -manifolds are cohomology n -manifolds. More examples can be found in [21].

Definition 2.2. A space X is called a *generalized n -manifold*, $n \geq 2$, if it is a finite-dimensional cohomology n -manifold.

If a generalized n -manifold X satisfies $H_c^n(X) \simeq \mathbb{Z}$, then X is said to be orientable and a choice of a generator g_X in $H_c^n(X)$ is called an orientation; X together with g_X is an oriented generalized n -manifold. If X is oriented, we can simultaneously choose an orientation g_U for all connected open subsets U of X via the isomorphisms

$$H_c^n(U) \rightarrow H_c^n(X).$$

Let X and Y be oriented generalized n -manifolds, $\Omega \subset X$ be an oriented domain and let $f: \Omega \rightarrow Y$ be continuous. For each domain $D \subset\subset \Omega$ and for each component V of $Y \setminus f(\partial D)$, the map

$$f|_{f^{-1}(V) \cap D} : f^{-1}(V) \cap D \rightarrow V$$

is proper. Hence we have a sequence of maps

$$(2.2) \quad H_c^n(V) \rightarrow H_c^n(f^{-1}(V) \cap D) \rightarrow H_c^n(D),$$

where the first map is induced by f and the second map is the standard homomorphism. The composition of these two maps sends the generator g_V to an integer multiple of the generator g_D ; this integer, denoted by $\mu(y, f, D)$, is called the *local degree of f at a point $y \in V$ with respect to D* . The local degree is an integer-valued locally constant function

$$y \mapsto \mu(y, f, D)$$

defined in $Y \setminus f(\partial D)$. If $V \cap f(D) = \emptyset$, then $\mu(y, f, D) = 0$ for all $y \in V$.

Definition 2.3. A continuous map $f: X \rightarrow Y$ between two oriented generalized n -manifolds is said to be *sense-preserving* if

$$\mu(y, f, D) > 0$$

whenever $D \subset\subset X$ is a domain and $y \in f(D) \setminus f(\partial D)$.

The following properties of the local degree can be found in [21].

Proposition 2.4 (Basic Properties of the Local Degree). (a) If $f, g: X \rightarrow Y$ are homotopic through proper maps h_t , $0 \leq t \leq 1$, such that $y \in Y \setminus h_t(\partial D)$ for all t , then

$$\mu(y, f, D) = \mu(y, g, D).$$

(b) If $y \in Y \setminus h_t(\partial D)$ and if $f^{-1}(y) \subset D_1 \cup \dots \cup D_p$, where D_i are all disjoint domains and contained in D such that $y \in Y \setminus f(\partial D_i)$, then

$$\mu(y, f, D) = \sum_{i=1}^p \mu(y, f, D_i).$$

(c) If $f: D \rightarrow f(D)$ is a homeomorphism, then $\mu(y, f, D) = \pm 1$ for each $y \in f(D)$. In particular, if f is a local homeomorphism, there is for each $x \in X$ a connected neighborhood D such that $\mu(f(x), f, D) = \pm 1$. More generally, if f is discrete and open and $x \in X$, then there is a relatively compact neighborhood D of x such that $\{f^{-1}(f(x))\} \cap \overline{D} = \{x\}$; the number $\mu(f(x), f, D) =: i(x, f)$ is independent of D and called the *local index of f at x* .

(d) If f is open, discrete, and sense-preserving, then for each $x \in X$ there is a connected neighborhood D as above such that $f(\partial D) = \partial f(D)$; D is called a *normal neighborhood* of x , and

$$(2.3) \quad i(x, f) = \max_{y \in f(D)} \text{card}\{f^{-1}(y) \cap D\}.$$

If D is any domain such that $f(\partial D) = \partial f(D)$, then D is called a *normal domain*.

2.2. Inverse dilatation. Let $f: X \rightarrow Y$ be continuous. For each $x \in X$, denote by $U(x, r)$ the component of x in $f^{-1}(B(f(x), r))$.

Set

$$H_f^*(x, s) = \frac{L_f^*(x, s)}{l_f^*(x, s)},$$

where

$$L_f^*(x, s) = \sup_{z \in \partial U(x, s)} d(x, z) \quad \text{and} \quad l_f^*(x, s) = \inf_{z \in \partial U(x, s)} d(x, z).$$

The *inverse linear dilatation function* of f at x is defined pointwise by

$$H_f^*(x) = \limsup_{s \rightarrow 0} H_f^*(x, s)$$

2.3. Doubling and Ahlfors regular metric spaces. A metric space X is called *doubling with constant N* , where $N \geq 1$ is an integer, if for each ball $B(x, r)$, every $r/2$ -separated subset of $B(x, r)$ has at most N points. We also say that X is *doubling* if it is doubling with some constant that need not be mentioned. It is clear that every subset of a doubling space is doubling with the same constant. A metric space X is called *locally doubling* if there is an integer $N > 0$ such that for each $x \in X$, there exists a ball $B(x, r)$ that is doubling with constant N .

A Borel regular measure μ on a metric space (X, d) is called a *doubling measure* if every ball in X has positive and finite measure and there exists a constant $C_\mu \geq 1$ such

that

$$(2.4) \quad \mu(B(x, 2r)) \leq C_\mu \mu(B(x, r))$$

for each $x \in X$ and $r > 0$. We call μ a *locally doubling measure* if there exists a constant $C_\mu \geq 1$ such that for each $x \in X$, there is a radius $r_x > 0$ with (2.4) holds for all $r \in (0, r_x)$.

A metric measure space (X, d, μ) is *Ahlfors Q -regular*, $1 \leq Q < \infty$, if there exists a constant $C \geq 1$ such that

$$(2.5) \quad C^{-1}r^Q \leq \mu(B(x, r)) \leq Cr^Q$$

for all balls $B(x, r) \subset X$ of radius $r < \text{diam } X$. It is well-known that if (X, d, μ) is an Ahlfors Q -regular space, then

$$(2.6) \quad \mu(E) \approx \mathcal{H}^Q(E)$$

for all Borel sets E in X ; see e.g. [15, Chapter 8]. A metric space X is called *locally Ahlfors Q -regular*, $1 \leq Q < \infty$, if there is a constant $C \geq 1$ such that for each $x \in X$, there exists a ball $B(x, r_x) \subset X$ that is Ahlfors Q -regular with constant C .

2.4. Loewner spaces. Let $X = (X, d, \mu)$ be a metric measure space and let Γ a family of curves in X . A Borel function $\rho: X \rightarrow [0, \infty]$ is *admissible for Γ* if for every locally rectifiable curves $\gamma \in \Gamma$,

$$(2.7) \quad \int_\gamma \rho \, ds \geq 1.$$

The p -modulus of Γ is defined as

$$\text{Mod}_p(\Gamma) = \inf \left\{ \int_X \rho^p \, d\mu : \rho \text{ is admissible for } \Gamma \right\}.$$

Definition 2.5. Let (X, d, μ) be a pathwise connected metric measure space. We call X a *Q -Loewner space* if there is a function $\phi: (0, \infty) \rightarrow (0, \infty)$ such that

$$\text{Mod}_Q(\Gamma(E, F, X)) \geq \phi(\zeta(E, F))$$

for every non-degenerate compact connected sets $E, F \subset X$, where

$$\zeta(E, F) = \frac{\text{dist}(E, F)}{\min\{\text{diam } E, \text{diam } F\}}.$$

By [18, Corollary 5.13], a complete (or equivalently proper) Ahlfors Q -regular metric measure space that supports a $(1, Q)$ -Poincaré inequality is Q -Loewner.

2.5. Density, porosity and generalized branch set. Let $S \subset X$ be a fixed set. We say S is δ -dense in $U \subset X$ if $U \subset \cup_{x \in S} B(x, \delta)$. We say S is δ -dense at x_0 , at scale R_0 , if S is δr -dense in $B(x_0, r)$ for each $r < R_0$. We also simply say S is δ -dense at x_0 , if it is δ -dense at some scale.

A set $E \subset X$ is said to be α -porous if for each $x \in E$,

$$\liminf_{r \rightarrow 0} r^{-1} \sup \{ \rho : B(z, \rho) \subset B(x, r) \setminus E \} \geq \alpha.$$

A subset E of X is called *countably (σ) -porous* if it is a countable union of (σ) -porous subsets of X .

Fix $x_0 \in X$, $y_0 = f(x_0)$, $r > 0$. We say a map $g: B(y_0, r) \rightarrow X$ is a *local left homotopy inverse* for f at x_0 if $g \circ f|_{U(x_0, r)}$ is homotopic to the identity on $U(x_0, r)$, via a homotopy H_t for which $x_0 \notin H_t(\partial U(x_0, r))$ for all t . Similarly, g is a *local right homotopy inverse* for f if $f \circ g$ is homotopic to the identity on $B(y_0, r)$, via a homotopy H_t with $y_0 \notin H_t(\partial B(y_0, r))$ for all t . If g is a left and right local homotopy inverse, we simply call it a *local homotopy inverse*.

We denote by \mathcal{B}_f^* the *homotopy branch set* of f , i.e. the set of points in X for which f has no (two-sided) local homotopy inverse. We also let $\mathcal{B}_f^{*,l}$ denote the *left homotopy branch set*, i.e. the set of points in X at which f has no left homotopy inverse. It is clear that if X and Y are generalized n -manifolds, then $\mathcal{B}_f = \mathcal{B}_f^{*,l}$.

3. BLD MAPPINGS BETWEEN LOEWNER SPACES ARE QUASIREGULAR

In the section, we take as standing assumptions that X and Y are two Ahlfors Q -regular, Q -Loewner, generalized n -manifolds. Under these assumptions, it follows from [18, Corollary 5.3] and [19, Theorem 7.3.2] that X and Y are quantitatively *quasiconvex*, i.e. each two points in the space can be joined by a curve whose length is at most a constant multiple the distance between these two points. Note that generalized manifolds of type A , considered by Heinonen and Rickman [21], are very special cases of metric spaces that satisfy our standing assumptions.

Our aim of this section is to show that BLD mappings between such spaces are quasiregular, quantitatively. Before stating our main result, let us recall first the definition of a BLD mapping.

Definition 3.1. A branched covering $f: X \rightarrow Y$ between two metric spaces is said to be an L -BLD, or a *mapping of L -bounded length distortion*, $L \geq 1$, if

$$L^{-1}l(\alpha) \leq l(f \circ \alpha) \leq Ll(\alpha)$$

for all non-constant paths α in X , where $l(\gamma)$ denotes the length of a curve γ in a metric space.

For a continuous mapping $f: X \rightarrow Y$ between two metric spaces, we set

$$L_f(x) = \limsup_{y \rightarrow x} \frac{d(f(x), f(y))}{d(x, y)} \quad \text{and} \quad l_f(x) = \liminf_{y \rightarrow x} \frac{d(f(x), f(y))}{d(x, y)}.$$

Proposition 3.2. Let $f: X \rightarrow Y$ be a branched covering. Consider for the following statements:

- 1). f is L -BLD;
- 2). For each $x \in X$, there exists $r_x > 0$ such that

$$\frac{d(x, y)}{c} \leq d(f(x), f(y)) \leq cd(x, y)$$

for all $y \in B(x, r_x)$;

- 3). $L_f(x) \leq c$ and $l_f(x) \geq \frac{1}{c}$ for each $x \in X$;

4). f is K -quasiregular, locally M -Lipschitz and $J_f(x) \geq c$ for a.e. $x \in X$.

We have $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4)$. Moreover, all the constants involved depend quantitatively only on each other and on the data associated to X and Y .

Proof. 1) implies 2): As pointed out in the beginning of this section, our standing assumptions on X implies that it is quasiconvex, quantitatively. Thus, for each $x, y \in X$, we may choose a quasiconvex curve $\gamma \subset X$ that joins x to y . Then

$$d(f(x), f(y)) \leq l(f(\gamma)) \leq Ll(\gamma) \leq CLd(x, y).$$

For the reverse inequality, fix a point $x \in X$ and we may work in a normal neighborhood U of x . Namely, consider $f: U \rightarrow B(f(x), r)$, where $B(f(x), r) = f(U)$. Since Y is quasiconvex, for each $x, y \in X$ with $f(y) \in B(f(x), r/C)$ and so we may fix a quasiconvex curve $\gamma' \subset Y$ that joins $f(x)$ to $f(y)$ in $B(f(x), r)$. By the path-lifting property of discrete and open mappings [37], we know there exists a curve $\gamma \subset X$ that joins x to y . Thus there exists $r_x > 0$ such that

$$d(x, y) \leq l(\gamma) \leq Ll(\gamma') \leq LCd(f(x), f(y))$$

for all $y \in B(x, r_x)$.

2) implies 3) is clear.

3) implies 4): 3) implies that f is locally c -Lipschitz and hence belongs to $N_{loc}^{1,Q}(X, Y)$ (see [19] for the definition). Moreover, since L_f is an *upper gradient* of f (see e.g. [19]),

$$g_f^Q(x) \leq L_f(x)^Q \leq c^{2Q} l_f(x)^Q,$$

where g_f is the *minimal Q -weak upper gradient* of f . Since X and Y are Ahlfors Q -regular,

$$l_f(x)^Q \leq C \limsup_{r \rightarrow 0} \frac{\mathcal{H}^Q(f(B(x, r)))}{r^Q} \leq C J_f(x)$$

for a.e. $x \in X$. This implies that f is *analytically C -quasiregular*. Since our metric spaces are Ahlfors Q -regular and Q -Loewner, by [14, Theorem A], analytically quasiregular mappings are quantitatively equivalent with (metrically) quasiregular mappings, and so 4) follows. □

Remark 3.3. i). It is clear that Proposition 3.2 2) implies that $H_f(x) \leq c^2$ for all $x \in X$ and hence f is metrically c^2 -quasiregular. In particular, this means that L-BLD mappings are always metrically H -quasiregular, quantitatively.

ii). If X and Y are the Euclidean spaces, then by [31, Theorem 2.16], we have 4) implies 1) as well. Thus Proposition 3.2 provides a quantitative characterization of BLD mappings in terms of quasiregular mappings. This characterization has been generalized to a greater generality in [21, Theorem 6.18], namely, for mappings from a generalized n -manifold of type A to \mathbb{R}^n . In our following up work [14], we have shown that such a characterization holds in a much wider situation.

4. QUANTITATIVE ENR THEORY FOR LINEARLY LOCALLY n -CONNECTED SPACES

In the topological setting, the way to prove that a locally compact, finite dimensional, separable and locally contractible space X is an *abstract neighborhood retract* (or ANR, for short), actually, an *Euclidean neighborhood retract* (or ENR, for short) (see [25] for definitions and general properties of ANR's and ENR's), is to first embed X with a proper map into some Euclidean space \mathbb{R}^n via *Whitney's embedding theorem*, and then construct a retraction $r: U \rightarrow X$ inductively on the k -skeletons of a *Whitney decomposition* of $U \setminus X$, where U is taken as a union of neighborhoods whose intersections with X are small enough to allow repeated applications of the local contractibility property. Then when $X \subset Y$, extending the embedding to a continuous map $f: Y \rightarrow \mathbb{R}^n$ gives us a retraction r from the neighborhood $f^{-1}(U)$ onto X .

Moreover, many of the general topological properties of an ANR X involve construction of homotopies between “close” maps into X . The method involves first embedding X into a locally convex topological vector space, and taking a retraction r from a neighborhood U . If the two maps are close enough so that the image of the linear homotopy between them lies in U , the composing the linear homotopy with $r \circ f|_{f^{-1}(U)}$ yields a homotopy entirely contained in X .

Thus, in the finite dimensional case, most of the important facts about ANR's can be obtained rather directly from the specific retraction that was constructed from the Whitney embedding theorem, and repeated applications of local contractibility. This partially motivates our approach.

In our setting, we suppose that X is separable, locally compact, locally doubling, and locally λ -LLC n . This implies X is an ANR, but we can in fact obtain more quantitative results. Since all of our considerations are local, we may ease the exposition by assuming that X is *precompact* (that is, its completion \overline{X} is compact), doubling, and that every ball $B \subset X$ is contractible in λB , provided that $\overline{\lambda B} \cap \partial X = \emptyset$; here $\partial X = \overline{X} \setminus X$ - note that by local compactness, X is open in \overline{X} .

We recall some basic results in quantitative topology. We essentially follow the logic of [34, 42], showing that close maps are homotopic by homotopies that don't move points very far - but we must use some care to ensure that individual points aren't moved too close to each other. Though our applications in the rest of the paper assume linear local contractibility, we give some of the results here in terms of linear local n -connectivity, in keeping with the spirit of [34].

Let X be a locally complete metric space, with completion \overline{X} and boundary $\partial X = \overline{X} \setminus X$. We say that X is λ -linearly locally n -connected (abbreviated λ -LLC n) if for each $x \in X$ and $r < 2d(x, \partial X)/\lambda$, the ball $B(x, r)$ is n -connected in $B(x, \lambda r/2)$. We say X is locally λ -LLC n if for each $x \in X$ there is a neighborhood $U \ni x$ that is λ -LLC n . When λ is unimportant, we omit it and say that X is *linearly locally n -connected* (-LLC n). We define λ -linear local contractibility (abbreviated λ -LLC *) in the same way as above, requiring instead that $B(x, r)$ be contractible inside $B(x, \lambda r/2)$.

Remark 4.1. Our definitions are quantitatively equivalent to the usual ones elsewhere in the literature. The factor of $1/2$ is included purely for convenience, as it implies

that if X is λ -LLC n , then for each $k \leq n + 1$, and each map $\sigma: \partial\Delta^k \rightarrow U$ with $\text{diam } \sigma(F) < d(\sigma(F), \partial X)/\lambda$ on each face $F \subset \partial\Delta^k$, σ extends to a map $\sigma': \Delta^k \rightarrow X$, with $\text{diam } \sigma'(\Delta^k) \leq \lambda \text{diam } \sigma(\partial\Delta^k)$.

Remark 4.2. The -LLC n condition is a stronger, quantitative form of local n -connectivity - the latter notion assumes each neighborhood of U of x has a smaller neighborhood $V \subset U$ that is n -connected in U ; λ -LLC n implies that if $U \supset B(x, r)$, then we may additionally take $V = B(x, 2r/\lambda)$. The same analogy holds likewise between local contractibility and -LLC * .

Remark 4.3. Recall that a metric space X is said to have λ -bounded turning if every pair $x_1, x_2 \in X$ may be joined with a continuum with diameter at most $\lambda d(x_1, x_2)$. Since local connectivity is equivalent to local path connectivity (under our standing local compactness assumption), it follows that if X has λ -bounded turning, then it is 2λ -LLC 0 , and conversely, if X is λ -LLC 0 , then it has λ -bounded turning.

Remark 4.4. We caution the reader that the -LLC 0 condition is sometimes denoted “-LLC $_1$ ”, and is only half of what is typically referred to in the literature as “*linear local connectivity*” or -LLC; -LLC also includes a dual assumption, sometimes called “-LLC $_2$ ”, that points outside $B(x, r)$ may be joined by a path lying outside of $B(x, r/\lambda)$. This can be thought of a quantitative version of X having no local cut points.

Many interesting spaces satisfy the -LLC $_2$ condition (e.g., Loewner spaces of dimension greater than 1). Moreover, without it, a few technical complications arise (see below). Despite this, we will typically not assume -LLC $_2$. The reason for this is that our most general results avoid the use of the strong analytic properties of Loewner spaces, and thus have potential to be applied to trees and other 1-dimensional spaces where the -LLC $_2$ condition may fail.

We need the following basic extension result, which follows by an induction on the k -skeleton. We suppose P is an n -dimensional simplicial complex, and $Q \subset P$ is a subcomplex containing P^0 . (The statement in [34] is slightly different, but the proof is the same.)

Lemma 4.5 ([34], Section 2, Main Lemma). Let X be λ -LLC $^{n-1}$, and let $\phi: Q \rightarrow X$ be a continuous map such that $\text{diam } \phi(\Delta \cap Q) < d(\phi(\Delta \cap Q), \partial X)/\lambda^n$ for each simplex $\Delta \subset P$. Then ϕ extends to a continuous map $\psi: P \rightarrow X$, such that for each simplex $\Delta \subset P$,

$$\text{diam } \psi(\Delta) \leq \lambda^n \text{diam } \phi(\Delta \cap Q).$$

Proposition 4.6. Suppose that Z is an ANR with $\dim(Z) \leq n$, that X is λ -LLC n , and that $g_0, g_1: Z \rightarrow X$ satisfy $d(g_0(z), g_1(z)) < d(\{g_0(z), g_1(z)\}, \partial X)/\lambda^{n+1}$, for each $z \in Z$. Then for each $\varepsilon > 0$, there is a homotopy $H: [0, 1] \times Z \rightarrow X$ such that for every $z \in Z$,

$$\text{diam } H([0, 1] \times \{z\}) \leq 4(1 + \varepsilon)\lambda^{n+1}d(g_0(z), g_1(z)).$$

Proof. We may with no loss of generality reduce to the case that $f(z) \neq g(z)$ for all $z \in Z$. Indeed, having proved this special case, applying the proposition to the restrictions $g_i|_{Z+}$,

where $Z^+ = \{z \in Z : g_0(z) \neq g_1(z)\}$, gives a homotopy that extends continuously to a constant homotopy on $\{z \in Z : g_0(z) = g_1(z)\}$.

In light of the aforementioned reduction, we let γ be an open covering of Z such that for each $V \in \gamma$,

$$\text{diam } g_0(V) + \text{diam } g_1(V) < (1 + \varepsilon/3)d(g_0(V), g_1(V)).$$

Let Q' be a dominating complex for γ , with $\dim(Q') = \dim(Z) \leq n$, i.e., there are maps $\rho: Q' \rightarrow Z$, $\iota: Z \rightarrow Q'$, such that $\rho \circ \iota$ is homotopic to the identity, via a homotopy $H^\gamma: I \times Z \rightarrow Z$ such that for each $\Delta' \subset Q'$, there is some $V \in \gamma$ for which $H^\gamma(I \times \iota^{-1}(\Delta')) \subset V$.

Now let $P = [0, 1] \times Q'$, $Q = \{0, 1\} \times Q' \subset P$. Define $\phi: Q \rightarrow X$ by $\phi(i, q) = g_i(\rho(q'))$. Note that P has a triangulation, where each simplex $\Delta \subset P$ lies inside $[0, 1] \times \Delta'$ for some $\Delta' \subset Q'$. It follows that

$$\begin{aligned} \text{diam } \phi(Q \cap \Delta) &\leq \text{diam } (g_0(\rho(\Delta')) \cup g_1(\rho(\Delta'))) \\ &\leq \text{diam } g_0(\rho(\Delta')) + \text{diam } g_1(\rho(\Delta')) + d(g_0(\rho(\Delta')), g_1(\rho(\Delta'))) \\ &\leq \text{diam } g_0(V) + \text{diam } g_1(V) + d(g_0(V), g_1(V)) \leq 2(1 + \varepsilon/3)d(g_0(V), g_1(V)) \end{aligned}$$

for some $V \in \gamma$ containing $H^\gamma(I \times \iota^{-1}(\Delta'))$. The extension $\psi: P \rightarrow X$ given by Lemma 4.5 therefore satisfies

$$\psi(\Delta) \subset B(\psi(v), 2(1 + \varepsilon/3)\lambda^n d(g_0(v), g_1(V)))$$

for each $v \in \Delta^0$, whereby we have

$$(4.1) \quad \text{diam } \psi(\Delta) \leq 2(1 + \varepsilon/3)\lambda^{n+1}d(g_0(V), g_1(V)).$$

Now, since $g_0(H^\gamma(z, 1)) = g_0(\rho(\iota(z))) = \psi(z, 0)$ and $g_1(H^\gamma(z, 1)) = g_1(\rho(\iota(z))) = \psi(z, 1)$, we may define a homotopy $H: Z \times [0, 1] \rightarrow X$ by

$$H(z, t) = \begin{cases} g_0(H^\gamma(3t, z)) & \text{if } 0 \leq t \leq \frac{1}{3} \\ \psi(3t - 1, \iota(z)) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3} \\ g_1(H^\gamma(3 - 3t, z)) & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$

Let $z \in Z$, with $\iota(z) \in \Delta' \subset Q'$, with $H^\gamma(I \times \iota^{-1}(\Delta')) \subset V \in \gamma$. Then

$$\begin{aligned} \text{diam } H(I \times \{z\}) &\leq \text{diam } g_0(H^\gamma(I \times \{z\})) + \text{diam } \psi(I \times \{z\}) + \text{diam } g_1(H^\gamma(I \times \{z\})) \\ &\leq \text{diam } g_0(V) + 2(1 + \varepsilon/3)\lambda^{n+1}d(g_0(V), g_1(V)) + \text{diam } g_1(V) \\ &\leq 4(1 + \varepsilon)\lambda^{n+1}d(g_0(V), g_1(V)) \leq 4(1 + \varepsilon)\lambda^{n+1}d(g_0(z), g_1(z)). \end{aligned}$$

□

Theorem 4.7. Let $A \subset X$, where X is λ -LLCⁿ and $\dim(A) \leq n$. Suppose that $f: X \rightarrow Y$ is a proper open map into some space Y with $\dim(Y) \leq n$, and that

$$\text{diam } f^{-1}(\{y\}) < d(f^{-1}(y), \partial X)/\lambda^{2n+1}$$

for each $y \in f(A)$, for some subset $A \subset X$. Then for each $\varepsilon > 0$, there is a map $g: f(A) \rightarrow X$ and a homotopy $H: I \times A \rightarrow X$, with $H_0 = id_A$, $H_1 = g \circ f|_A$, and

$$\text{diam } H(I \times \{x\}) \leq 8\lambda^{2n+1} \text{diam } f^{-1}(\{f(x)\})$$

for all $x \in A$.

Proof. The properness and openness of f imply that the function $y \mapsto \text{diam } f^{-1}(\{y\})$ is continuous (as a fortiori is the map $x \mapsto \text{diam } f^{-1}(\{f(x)\})$), so upon replacing A itself with a small neighborhood of A , the assumptions of the theorem remain valid. Thus we lose no generality supposing A is open, as is $f(A)$ by the openness of f .

Then A is an ANR, so by Proposition 4.6, it suffices to show for each $\varepsilon > 0$ there is a map $g: f(A) \rightarrow X$ such that

$$d(g(f(x)), x) < 2(1 + \varepsilon)\lambda^n \text{diam } f^{-1}(\{f(x)\})$$

for each $x \in A$. We may likewise assume that $\text{diam } f^{-1}(\{y\}) > 0$ for all $y \in Y$, since we may apply that case to the restriction $f|_{X^+}: X^+ \rightarrow Y$, where $X^+ = \{x \in X : \text{diam } f^{-1}(\{y\}) > 0\}$, and then extend g from $f(X^+ \cap A)$ to all of $f(A)$ by setting $g(y) = x$ whenever $f^{-1}(\{y\}) = \{x\}$.

To construct g , we first cover $f(A)$ with an open cover γ so that for each $V \in \gamma$ and each $y \in V$,

$$(4.2) \quad \text{diam } f^{-1}(V) \leq (1 + \varepsilon) \text{diam } f^{-1}(\{y\}).$$

This can be done, again, by the continuity and positivity of $y \mapsto \text{diam } f^{-1}(\{y\})$.

Now, taking P to be the nerve of a suitable refinement of γ , with $\dim(P) = \dim f(A) \leq n$, we obtain via a partition of unity a map $\iota: f(A) \rightarrow P$, such that for every simplex $\Delta \subset P$, there is some $V \in \gamma$ with $\iota^{-1}(\Delta) \subset V$. By taking the refinement of γ to be minimal, we may also assume that $\iota(f(A)) \supset P^0$.

Now, define $\rho^0: P^0 \rightarrow X$ so that $\rho^0(p_0) \in f^{-1}(\iota^{-1}(\{p_0\}))$, for each $p_0 \in P^0$. Then for each $\Delta \subset P$,

$$\rho^0(\Delta \cap P^0) \subset f^{-1}(\iota^{-1}(\Delta)) \subset f^{-1}(V)$$

for some $V \in \gamma$. We therefore have

$$\text{diam } \rho^0(\Delta \cap P^0) \leq \text{diam } f^{-1}(V),$$

so that the extension of ρ^0 to $\rho: P \rightarrow X$ from Lemma 4.5 satisfies

$$\text{diam } \rho(\Delta) \leq \lambda^n \text{diam } f^{-1}(V).$$

We now define $g = \rho \circ \iota: f(A) \rightarrow X$. To see that g satisfies the conclusion of the theorem, we suppose $x \in A$, and let $y = f(x)$, $p = \iota(y) \in \Delta$ for some simplex $\Delta \subset P$. Let $p' \in \Delta^0$ and let $x' = \rho(p')$, $y' = f(x')$. By our selection of ρ^0 , we have $g(f(x')) = x$. On the one hand, choosing some $V \in \gamma$ containing $\iota^{-1}(\Delta)$, we have that $x, x' \in f^{-1}(\iota^{-1}(\Delta)) \subset f^{-1}(V)$, so

$$d(x, x') \leq \text{diam } f^{-1}(V).$$

On the other hand, since $p, p' \in \Delta$,

$$d(g(f(x)), x') = d(\rho(p), \rho(p')) \leq \text{diam } \rho(\Delta) \leq \lambda^n \text{diam } f^{-1}(V),$$

and so combining the preceding inequalities with (4.2) gives

$$\begin{aligned} d(x, g(f(x))) &\leq (1 + \lambda^n) \text{diam } f^{-1}(V) \\ &\leq (1 + \varepsilon)(1 + \lambda^n) \text{diam } f^{-1}(\{f(x)\}) \\ &\leq 2(1 + \varepsilon)\lambda^n \text{diam } f^{-1}(\{f(x)\}). \end{aligned}$$

□

As a corollary of Theorem 4.7, we obtain the following generalized version of the McAuley-Robinson theorem [32], which is of independent interest.

Corollary 4.8. Let $A \subset X$, where X is a λ -LLCⁿ generalized n -manifold and $\dim(A) \leq n$ and let Y be another homology n -manifold. Let $f: X \rightarrow Y$ be a proper branched covering such that for some $x_0 \in A \setminus \partial A$, $f^{-1}(\{f(x_0)\}) = x_0$ and

$$\sup_{x \in \partial A} \frac{\text{diam } f^{-1}(\{f(x)\})}{d(x, x_0)} < \frac{1}{8\lambda^{2n+1}}.$$

Then $x_0 \notin \mathcal{B}_f$.

Proof. We have for some $\varepsilon > 0$ with $\text{diam } f^{-1}(\{f(x)\}) < d(x, x_0)/(8(1 + \varepsilon)\lambda^{2n+1})$ for each $x \in \partial A$. The homotopy H from the conclusion of Theorem 4.7 satisfies

$$H(I \times \{x\}) \subset B(x, d(x, x_0)) \not\ni x_0$$

for each $x \in \partial A$, whereby $x_0 \notin H(I \times \partial A)$. Thus $g \circ f$ has local degree 1 at x_0 , and so $i(x_0, f) = 1$, whence $x_0 \notin \mathcal{B}_f$. □

As a corollary of Corollary 4.8, we obtain a general strategy for proving nonbranching.

Corollary 4.9. Let X, Y, f be as in Corollary 4.8, and suppose $A \subset X$ with

$$\text{diam } A \leq \frac{d(A, \partial X)}{8\lambda^{2n+1}}$$

and $x_0 \in S \cap (A \setminus \partial A)$. Suppose further there is a subset $S = f^{-1}(f(S)) \subset X$, such that $f|_S$ is injective, and $\{U_\alpha\}$ is a family of connected open subsets such that $\partial A \subset \cup_\alpha U_\alpha$ and such that for each α ,

$$\partial f(U_\alpha) = f(\partial U_\alpha), \quad U_\alpha \cap S \neq \emptyset \quad \text{and} \quad \text{diam } U_\alpha < d(x_0, U_\alpha)/\lambda^{2n+1}$$

Then $x_0 \notin \mathcal{B}_f$.

Proof. Since $U_\alpha \cap S \neq \emptyset$ and $f(\partial U_\alpha) = \partial f(U_\alpha)$, we have that $U_\alpha = f^{-1}(f(U_\alpha))$, for each α . Since these sets cover ∂A , we obtain that for each $x \in \partial A$, $f^{-1}(\{f(x)\}) \subset U_\alpha$ for some α , and so

$$\text{diam } f^{-1}(\{f(x)\}) < d(x_0, f^{-1}(\{f(x)\}))/(\lambda^{2n+1}).$$

Applying Corollary 4.8 completes the proof. □

Remark 4.10. All of the results in this section generalize easily to spaces with local geometric connectivity or contractibility - the so called $\text{LGC}^n(\rho)$ - and $\text{LGC}^*(\rho)$ -spaces introduced by Gromov [8, 9, 10]. In fact, the arguments from [34] that we have modified were stated in that generality.

5. ANNULAR DISTORTION

In order to construct a homotopy inverse, we need to control the distortion of annuli, rather than simply that of spheres. This is not hard to do in the special case that f is quasiregular (i.e. $H_f(x)$ is finite and essentially bounded), provided X and Y are Loewner spaces.

The purpose of this section is to show that in the general case, this can be done, quantitatively, at points with finite dilatation, away from a porous set. Throughout this section, we assume that X and Y are locally doubling and have bounded turning, that $f: X \rightarrow Y$ is a branched covering, and that $S = S_{H,R} = \{x \in X : H_f(x, r) < H \text{ for all } r < R\}$. Note that it follows from the definition that for each $x \in S$, $f|_{S \cap B(x, R/2)}$ is injective, and that $S \cap B(x, R/2) = f^{-1}(f(S \cap B(x, R/2))) \cap B(x, R/2)$.

Since Y has bounded turning, it follows that the new metric $d'(y_1, y_2) := \inf_{\gamma} \text{diam } \gamma$, where the infimum is taken over all continua γ in Y joining y_1 and y_2 , is C -bilipschitz equivalent to the original metric d on Y , where C is the constant of bounded turning. Thus we lose no generality in our considerations if we post-compose f with this change of metric, as the linear dilatation is increased by at most a factor of C^2 .

This reduction has the convenience that if Y has 1-bounded turning, then for each $x' \in X$, $r' > 0$, we have

$$(5.1) \quad B(f(x'), l_f(x', r')) \subseteq f(B(x', r')),$$

by the path lifting property of discrete open maps [6, Theorem 3]. Moreover, if $f^{-1}(\{y\}) \cap U = \{x\}$, for some connected normal neighborhood U of x containing $f(B(x, r))$, then we have

$$(5.2) \quad f^{-1}(B(y, l_f(x, r'))) \cap U = U(x, l_f(x, r')) \subseteq B(x, r').$$

In light of the above reduction, we assume here-on-out that Y has 1-bounded turning.

In the ensuing Lemma, we use the convention that $y = f(x)$, $y' = f(x')$, etc.

Lemma 5.1. Suppose $x \in S$ and $0 < r < R$. For each $x' \in B(x, r)$, we have $d(y', y) < HL_f(x, r)$. That is $d(y', y) < H^2 d(y'', y)$, whenever $d(x', x) < d(x'', x) < R$.

Proof. The lemma follows immediately from the observation that $l_f(x, r') < L_f(x, r)$ whenever $r' < r$. To see that this inequality holds, note that if $L_f(x, r) \leq l_f(x, r')$, then by the inclusion (5.2), we have $S(x, r) \subseteq f^{-1}(B(y, l_f(x, r'))) \cap U \subseteq B(x, r')$ (here $S(x, r)$ denotes the sphere of radius r), whereby $r \leq r'$. \square

In the ensuing propositions and lemmas, we always suppose that $x_0 \in S$ and $y_0 = f(x_0)$.

Proposition 5.2. For each $\lambda \geq 1$, there is a constant $C_\lambda > 1$, depending only on λ and the data, such that if $f(S)$ is $r/3$ -dense in $B(y_0, C_\lambda r)$, then $l_f^*(x_0, C_\lambda r) \geq \lambda L_f^*(x_0, r)$.

Proof. Suppose that $f(S)$ is $r/3$ -dense in $B(y_0, C_\lambda r)$, where $C_\lambda > H^2 + H$ has yet to be determined. Let $M = \lfloor \frac{\log(C_\lambda/H)}{2\log(H+1)} \rfloor$ so that

$$(5.3) \quad (H+1)^{2M} \leq C_\lambda/H \leq (H+1)^{2M+2}.$$

Since Y is connected, it follows that

$$(5.4) \quad f(S) \cap \left(B(y, (H+1)^{2k}r) \setminus B(y, (H+1)^{2k-1}r) \right) \neq \emptyset,$$

for $k = 1, \dots, M$. Because $l_f(x_0, l_f^*(x_0, C_\lambda r)) \geq L_f(x_0, l_f^*(x_0, C_\lambda r))/H = C_\lambda r/H$, we may therefore choose a sequence of points y_1, \dots, y_M with

$$y_k \in f\left(S \cap B(x_0, l_f(x_0, l_f^*(x_0, C_\lambda r)))\right) \cap \left(B(y_0, (H+1)^{2k}r) \setminus B(y_0, (H+1)^{2k-1}r) \right)$$

It follows from the triangle inequality that $d(y_j, y_k) > Hd(y_0, y_j)$ for each $j < k$.

Choosing $x_k \in S \cap l_f(x_0, l_f^*(x_0, C_\lambda r)) \cap f^{-1}(\{y_k\})$, Lemma 5.1 implies that

$$d(x_j, x_k) > d(x_0, x_j) \geq L_f^*(x_0, r),$$

with the second inequality coming from the fact that for each $x \in U(x_0, r)$, and each j , we have $d(y_0, y_j) > Hd(y_0, y)$. Thus $\{x_k : 0 \leq k \leq M\}$ is an $L_f^*(x_0, r)$ -separated subset of $B(x_0, l_f(x_0, l_f^*(x_0, C_\lambda r)))$ with M elements, and so

$$M \leq C \left(\frac{l_f^*(x_0, C_\lambda r)}{L_f^*(x_0, r)} \right)^s,$$

where C and s depend only on the doubling constant of X .

We therefore let $C_\lambda = H(H+1)^{2(C\lambda^s+1)}$, so that $M = \lfloor C\lambda^s + 1 \rfloor \geq C\lambda^s$, and the proposition is proved. \square

We need an analogue of Proposition 5.2 when S is dense at x_0 . For this, we require a slight strengthening of a Lemma of Tukia and Väisälä [43, Theorem 2.9].

Lemma 5.3. Let X be doubling and have C_0 -bounded turning. There there is some integer $M \in \mathbb{N}$, depending only on the data of X , such that every pair of points $x, x' \in X$ with $d(x, x') = 8r$ may be joined with a sequence $x = x_0, \dots, x_k, \dots, x_N = x'$, $N \leq M$, of points in $B(x, C_0 d(x, x'))$, such that $d(x_0, x_1) = 2r/3$, and

$$d(x_j, x_{j+1}) \leq d(x_{j-1}, x_j) - \frac{r}{3M},$$

with equality whenever $j < N - 1$.

Proof. Choose C so that every $r/3$ -separated subset of every ball $B(x, 8C_0 r)$ has at most M members.

Now, given $x, x' \in X$ with $d(x, x') = 8r$, join x to x' in $B(3Cr)$ by a path $\gamma: [0, 1] \rightarrow B(x, 8C_0 r)$.

We shall choose values $\{t_j\} \subset [0, 1]$ for each $j \leq M$, and let $x_j = \gamma(t_j)$. We do this inductively as follows. Let $\gamma(t_0) = x_0 = x$, let $\gamma(t_1)$ is the largest possible value

for which $d(x_1, x_0) \leq 2r/3$, and for $k > 2$, t_k is the largest possible value such that $d(x_k, x_j) \leq \max\{d(x_j, x_{j-1}) - \frac{r}{3M}, 0\}$ for some $j < k$.

Notice that by continuity, $d(x_1, x_0) = r$, and for $k > 1$, either $x_k = x'$, or $d(x_k, x_j) = \max\{d(x_j, x_{j-1}) - \frac{r}{3M}, 0\}$ for some $j < k$, and $d(x_k, x_l) \geq \max\{d(x_l, x_{l-1}) - \frac{r}{3M}, 0\}$ for each $l < j$. Removing intermediate points, we may take $k = j + 1$.

Now, either the points x_k , $k = 1, \dots, M$ are distinct, or $x_k = x'$ for some $k \leq M$. In the latter case, we are done, taking N to be the first index for which $x_N = x'$. Suppose then that the points are distinct. By construction, then, they satisfy $d(x_{j+1}, x_j) = 2r/3 - \frac{jr}{3M} > r/3$, so that they form an $r/3$ -separated set in $B(x, 8C_0r)$, with $M + 1$ members, contradicting our choice of M . \square

Proposition 5.4. For each $\lambda > 1$, if S is $r/(6M)$ -dense in $B(x_0, 8C_0r)$, then

$$L_f(x_0, 8r) \leq H^{M+2}l_f(x_0, r),$$

where C_0 is the constant of bounded turning for X and M is the constant from Lemma 5.3.

Proof. Suppose $d(x, x') = 8r$. Choose a sequence x_k , $k = 1, \dots, N$, $N \leq M_\lambda$ as in the conclusion of Lemma 5.3. By the $r/3$ -density of S in $B(x, 8C_0r)$, we may choose for $k = 1, \dots, N - 1$ points $x'_k \in S \cap B(x_k, r/3)$. Setting $x'_0 = x_0$ and $x'_N = x_N$, we then obtain a sequence x'_0, \dots, x'_N with $d(x'_{k+1}, x'_k) \leq d(x'_k, x'_{k-1})$. It follows from Lemma 5.1 that $d(y'_{k+1}, y'_k) \leq Hd(y'_k, y'_{k-1})$ for each k , so that

$$\begin{aligned} d(y', y_0) &\leq \sum_{j=1}^N d(y'_j, y'_{j-1}) \leq \sum_{j=1}^N H^{j-1}d(y_1, y_0) = \left(\frac{H^N - 1}{H - 1}\right)d(y_1, y_0) \\ &\leq \left(\frac{H^M - 1}{H - 1}\right)d(y'_1, y_0) \leq H^2 \left(\frac{H^M - 1}{H - 1}\right)l_f(x_0, r), \end{aligned}$$

with the last inequality resulting from Lemma 5.1 and the fact that

$$d(x'_1, x_0) \leq d(x_0, x_1) + d(x_1, x'_1) \leq 2r/3 + r/6 < r.$$

Since this holds for $y' = f(x')$ for arbitrary x' in $S(x_0, 8r)$, the proof is complete. \square

Proposition 5.5. For each $\lambda > 1$, there is a constant $C'_\lambda > 1$, depending only on λ and the data, such that if S is $r/(6M)$ -dense in $B(x_0, 2C_0C'_\lambda r)$, then

$$l_f(x_0, C'_\lambda r) \geq \lambda L_f(x_0, r).$$

Proof. We take $C'_\lambda = 4^N$, for some N to be determined. Choose points $x_k \in S \cap (B(x_0, 4^k r) \setminus B(x_0, 2^{2k-1} r))$ for $k = 1, \dots, N$. By Proposition 5.4, we have

$$L_f(x_k, 2^{2k+1} r) \leq H^{M+2}l_f(x_k, 2^{2k-1} r)$$

for each k . Since $B(x_0, r) \subset B(x_k, 2^{2k+1} r)$, we then have

$$L_f(x_0, r) \leq \text{diam } f(B(x_k, 2^{2k+1} r)) \leq 2HL_f(x_k, 2^{2k+1} r) \leq 2H^{M+1}l_f(x_k, 2^{2k-2} r).$$

Since $x_j \notin B(x_k, 2^{2k-2}r)$ for $j \neq k$, we have $d(y_j, y_k) \geq l_f(x_k, 2^{2k-2}r)$. Thus $\{y_k\}$ is an $\frac{L_f(x_0, r)}{2^{H+3}}$ -separated subset of $f(B(x, C''_\lambda r))$, whereby

$$N \leq C \left(\frac{2^{H+3} \text{diam } f(B(x_0, C'_\lambda r))}{L_f(x_0, r)} \right)^s \leq C \left(\frac{4^{H+5} l_f(x_0, C'_\lambda r)}{L_f(x_0, r)} \right)^s,$$

so that

$$L_f(x_0, r) \leq C N^{-1/s} l_f(x_0, C'_\lambda r),$$

with C depending only on H and the data of X and Y . Taking $N = \lceil (\lambda C)^s \rceil + 1$ completes the proof. \square

Corollary 5.6. Under the assumption of Proposition 5.5, there is some $s > 0$ such that

$$B(x_0, r) \subset U(x_0, s) \subset B(x_0, C'_1 r).$$

6. PROOFS OF THE MAIN RESULTS

Proposition 6.1. Suppose X is LLC^n , and that $f: X \rightarrow Y$, $S \subset X$, H , and R are as in the previous section. Then there are constants $\varepsilon, \delta > 0$ depending only on H and the data of X and Y so that if either S is δ -dense at x_0 , or $f(S)$ is ε -dense at $y_0 = f(x_0)$, then $x_0 \notin \mathcal{B}_f^{*,l}$.

Proof. If $f(S)$ is εr -dense in $B(y_0, 3r)$, then there is a family of balls $B(y_\alpha, \varepsilon r)$ such that $\cup_\alpha B(y_\alpha, \varepsilon r) \supset \partial U(x_0, r)$. Let $U(x_\alpha, \varepsilon r)$ be the x_α -component of $f^{-1}(B(y_\alpha, \varepsilon r))$, with $x_\alpha \in S$. Then $\cup_\alpha U(x_\alpha, \varepsilon r) \supset \partial U(x_0, r)$. If ε is small enough, then we may assume also that $y_0 \notin B(y_\alpha, 16\lambda^{2n+1}\varepsilon r)$, so that

$$x_0 \notin B(x_\alpha, 16\lambda^{2n+1}L_f^*(x_\alpha, \varepsilon r)) \supset B(x_\alpha, 8\lambda^{2n+1} \text{diam } U(x_\alpha, \varepsilon r)),$$

whereby

$$\text{diam } U(x_\alpha, \varepsilon r) \leq d(x_0, U(x_\alpha, \varepsilon r)) / (8\lambda^{2n+1}).$$

It follows from Corollary 4.9 that $x_0 \notin \mathcal{B}_f^{*,l}$.

We argue similarly for the case that S is δr -dense in $B(x_0, 3Cr)$. In this case, we choose balls $B(x_\alpha, \delta r)$ covering $\partial B(x_0, r)$. Assuming that balls each intersect $\partial B(x_0, r)$, we may choose δ small enough so that

$$\text{diam } B(x_\alpha, C'_1 \delta r) \leq d(x_0, B(x_\alpha, C'_1 \delta r)) / (8\lambda^{2n+1}).$$

By Corollary 5.6, we have some $s_\alpha > 0$ such that $B(x_\alpha, \delta r) \subset U(x_\alpha, s_\alpha) \subset B(x_\alpha, C'_1 \delta r)$. Applying Corollary 4.9 with $U_\alpha = U(x_\alpha, s_\alpha)$ proves again that $x_0 \notin \mathcal{B}_f^{*,l}$.

Lastly, in the case that $H_f^*(x) \leq H$ for each $r' < R'$ and $x \in S$, we repeat the argument from the previous paragraph, this time choosing $s_\alpha = L_f(x_\alpha, \delta r)$, so that $B(x_\alpha, \delta r) \subset U(x_\alpha, s_\alpha) \subset B(x_\alpha, H\delta r)$. Choosing δ small enough so that

$$\text{diam } B(x_\alpha, H\delta r) \leq d(x_0, B(x_\alpha, H\delta r)) / (8\lambda^{2n+1})$$

to complete the proof. \square

Proof of Theorem 1.1. Let $S_{H,R} = \{x \in X : H_f(x, r) < H \text{ for all } r < R\}$. If $x_0 \in S_{H,R} \cap \mathcal{B}_f^{*,l}$, then by Proposition 6.1, $S_{H,R}$ is not δ_H -dense at x_0 . Thus $S_{H,R} \cap \mathcal{B}_f^{*,l}$ is δ_H -porous, with δ_H depending only on H and the data of X and Y . The same argument proves that $f(S_{H,R} \cap \mathcal{B}_f^{*,l})$ is ε_H -porous, with ε_H depending only on H and the data of X and Y . \square

7. VÄISÄLÄ'S INEQUALITY

In this section, we prove the Väisälä's inequality in general metric spaces. In the Euclidean setting, this inequality has been first proved by Väisälä [45], and it plays an important role in the value distributional results for quasiregular mappings between Euclidean spaces; see for instance the Zorich–Gromov global homeomorphism theorem [11, 24], the Bloch's theorem [35], the Rickman–Picard theorem [38], and the defect relation [39].

Let $f: X \rightarrow Y$ be a proper discrete open mapping between Ahlfors Q -regular, LLC metric spaces, such that $H_f(x) < \infty$ everywhere on X , and $H_f(x) \leq H$ for \mathcal{H}^Q -almost every $x \in X$, and suppose $N_f(x) < \infty$. For each family Γ of curves in X , we have the K_O - and K_I -inequalities

$$(7.1) \quad \text{Mod}_Q(f(\Gamma))/K_I \leq \text{Mod}_Q(\Gamma) \leq K_O N_f \text{Mod}_Q(f(\Gamma)).$$

The right hand of these inequalities was established by Cristea [5], and more recently, the left hand has been established by the second author in [47] (see also [14, Theorem A]). Moreover, in the case that the image of the branch set is zero, the left hand inequality (the Poletsky's inequality) may be upgraded as follows:

Definition 7.1 (Väisälä's inequality). We say that f satisfies *Väisälä's inequality* with constant K_I if it satisfies the following condition: Suppose $m \in \mathbb{N}$, and Γ and Γ' are curve families in X and Y respectively, such that for each $\gamma' \in \Gamma'$, there are curves $\gamma_1, \dots, \gamma_m \in \Gamma$ such that $f(\gamma_k)$ is a subcurve of γ' for each k , and for each $t \in [0, l(\gamma)]$ and each $x \in X$, we have $\text{card}\{k : \gamma_k(t) = x\} \leq i(x, f)$. Then

$$\text{Mod}_Q(\Gamma') \leq K_I \text{Mod}_Q(\Gamma)/m.$$

Theorem 7.2 (Theorem A, [14]). Suppose $f: X \rightarrow Y$ is a discrete open mapping between two metric measure spaces (X, μ) and (Y, ν) , such that for some K_I , f satisfies the Poletsky's inequality

$$\text{Mod}_Q(f(\Gamma)) \leq K_I \text{Mod}_Q(\Gamma)$$

for every curve family Γ in X , and such that $\nu(f(\mathcal{B}_f)) = 0$. Then f satisfies Väisälä's inequality with the same constant K_I .

Combining Theorem 7.2 and Corollary 1.2 with the first half of inequality (7.1), we obtain the following very general Väisälä's inequality.

Theorem 7.3 (Väisälä's inequality). Let X and Y be Ahlfors Q -regular generalized n -manifolds, where X is LLC^n and Y is LLC, and suppose $f: X \rightarrow Y$ is discrete and open, with $H_f(x) < \infty$ for all $x \in X$, and $H_f(x) \leq H$ for \mathcal{H}^Q -almost every $x \in X$.

Then f satisfies Väisälä's inequality for some constant K_I depending only on H and the data of X and Y .

A path-lifting theorem of Rickman [37] implies the following corollary.

Corollary 7.4. Let X , Y , and f be as in Theorem 7.3, and suppose $U \subset X$ is a normal domain. Then for every curve family Γ' in $f(U)$,

$$\text{Mod}_Q(\Gamma') \leq K_I \text{Mod}_Q(\Gamma)/N_f(U).$$

As a consequence of Corollary 7.4, we may generalize Theorem 1.4 with the exact argument from [30].

Proof of Theorem 1.4. The argument is the same as that of [30, Theorem 5.2]. We repeat the argument here for the reader's convenience, and to assure that everything generalizes as appropriate.

Fix $x \in X$, $y = f(x)$. Let $r > 0$, $L^* = L_f^*(x, r)$, $L = L_f(x, L^*)$, $l^* = l_f^*(x, r)$ and $l = l_f(x, L^*)$. Abbreviate $U_s = U(x, s)$, and $B_s = B(x, s)$. We may assume that r is chosen small enough so that f is injective on $\{x \in X : i(x, f) = i(x_0, f)\} \cap U_L$. Let Γ_1 be the family of curves joining U_l to $\Omega \setminus U_r$ and Γ_2 the family of curves joining U_r to $\Omega \setminus U_L$. Note that ∂U_l and ∂U_r meet $S(x, l^*)$, and that ∂U_r and ∂U_L meet $S(x, L^*)$. The Loewner condition implies that

$$\min\{\text{Mod}_Q(\Gamma_1), \text{Mod}_Q(\Gamma_2)\} \geq a > 0.$$

By the K_O -inequality,

$$\text{Mod}_Q(\Gamma_1) \leq cK_O i(x, f) \left(\log \frac{r}{l}\right)^{1-Q},$$

and

$$\text{Mod}_Q(\Gamma_2) \leq cK_O i(x, f) \left(\log \frac{L}{r}\right)^{1-Q}.$$

On the other hand, Väisälä's inequality from Corollary 7.4 gives

$$\begin{aligned} K_I \text{Mod}_Q(\Gamma(U_l, \Omega \setminus U_L)) &\geq i(x, f) \text{Mod}_Q(\Gamma(f(U_l), f(\Omega \setminus U_L))) \\ &\geq ci(x, f) \left(\log \frac{L}{l}\right)^{1-Q}. \end{aligned}$$

Note also that

$$\text{Mod}_Q(\Gamma(U_l, \Omega \setminus U_L)) \leq c \left(\log \frac{L^*}{l^*}\right)^{1-Q}.$$

The claim follows by combining the above estimates. \square

Remark 7.5. Our argument here for the dimension estimate in Corollary 1.5 is somewhat more direct than the one from [3] - by obtaining an index-independent porosity result, we avoided the need to show the slightly stronger result of porosity for the set of all branch points of high index, as was done in [3]. We believe that with Väisälä's inequality, Theorem 7.3, the arguments in [3] likely generalize, with some care, but we leave such investigations to the interested reader, as they are somewhat technical, and unnecessary for the dimension estimate.

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